# Deriving eq 8.3

The joys of trig

Consider a plane wave arriving from a direction $(\phi, \theta)$ in spherical coordinates. Think of the plane wave as a 2D surface extending infinitely in all directions, it's travelling along it's normal vector, which I'll call $\vec{w}$.

The time that plane wave would take to pass between two points $\vec{x_i}$, $\vec{x_j}$ is $\frac{1}{c}\times$ the component of the distance between the two points parallel to the plane wave normal vector $\vec{w}$.

In other words, it's a dot product.

$$\delta t^{ij}_{expected} = \frac{1}{c}\left(\vec{w} \cdot \vec{x_i} - \vec{x_j}\right) \tag{1}$$

What follows is pretty dull and nowhere near as important as getting a good physical intution for what I described above. But let's expand out the vector into cartesian components:

$$\vec{w} \cdot (\vec{x_i} - \vec{x_j}) = x_w(x_i - x_j) + y_w(y_i - y_j) + z_w(z_i - z_j) \tag{2}$$

Now we're going to do some coordinate substitutions. Since $\vec{w}$ can come in from any direction $(\phi, \theta)$ let's use spherical coordinates.

$$\begin{aligned} \mathrm{x}_w &= \cos\theta\cos\phi \\ \mathrm{y}_w &= \cos\theta\sin\phi \\ \mathrm{z}_w &= -\sin\theta \end{aligned} \tag{3}$$

where I've omitted the radial scale factor of 1 since the normal vector $\vec{w}$ has unit length, $\theta$ is the elevation angle: at 0 deg all points lie on the x-y plane, the -z axis is lies at +90 deg, and the +z axis lies −90 deg. (ANITA had a funky $\theta$ definition... it always used to confuse me.)

ANITA uses cylindrical coordinates to describe its antenna positions since the experiment has approximate cylindrical symmetry, so the $\vec{x_i}$ components can be expressed:

$$
\begin{aligned}
\mathrm{x}_i &= r_i \cos\phi_i \\
\mathrm{y}_i &= r_i \sin\phi_i \\
\mathrm{z}_i &= z_i
\end{aligned} \tag{4}
$$

Plugging these into equation 2, it gets messy

$$
\begin{aligned}
\vec{w}\cdot(\vec{x_i}-\vec{x_j}) =& + \cos\theta\cos\phi(r_i\cos\phi_i - r_j\cos\phi_j) \\
& + \cos\theta\sin\phi(r_i\sin\phi_i - r_j\sin\phi_j) \\
& - \sin\theta(z_i - z_j) \\
=& + r_i\cos\phi_i\cos\theta\cos\phi - r_j\cos\phi_j\cos\theta\cos\phi \\
& + r_i\sin\phi_i\cos\theta\sin\phi - r_j\sin\phi_j\cos\theta\sin\phi \\
& - \sin\theta(z_i - z_j)
\end{aligned} \tag{5}
$$

Now we collect up terms 1,3 and 2,4...

$$
\begin{aligned}
\vec{w}\cdot(\vec{x_i}-\vec{x_j}) =& + r_i\cos\theta(\cos\phi_i\cos\phi + \sin\phi_i\sin\phi) \\
& - r_j\cos\theta(\cos\phi_j\cos\phi + \sin\phi_j\sin\phi) \\
& - \sin\theta(z_i - z_j)
\end{aligned} \tag{6}
$$

Those terms in brackets in equation 6 are trigonometric identities, which we can plug in

$$
\begin{aligned}
\vec{w}\cdot(\vec{x_i}-\vec{x_j}) =& + r_i\cos\theta\cos(\phi_i - \phi) \\
& - r_j\cos\theta\cos(\phi_j - \phi) \\
& - \sin\theta(z_i - z_j) \\
=& + \cos\theta(r_i\cos(\phi_i - \phi) - r_j\cos(\phi_j - \phi)) \\
& - (z_i\sin\theta - z_j\sin\theta)
\end{aligned} \tag{7}
$$

We can use the relation $\tan\theta = \frac{\sin\theta}{\cos\theta}$ to pull a factor of $\cos\theta$ outside of the z-component terms.

$$
\begin{aligned}
\vec{w}\cdot(\vec{x_i}-\vec{x_j}) =& + \cos\theta(r_i\cos(\phi_i - \phi) - r_j\cos(\phi_j - \phi)) \\
& - \cos\theta(z_i\tan\theta - z_j\tan\theta)
\end{aligned} \tag{8}
$$

$$
\vec{w}\cdot(\vec{x_i}-\vec{x_j}) = \cos\theta\left[(z_j\tan\theta - r_j\cos(\phi_j - \phi)) - (z_i\tan\theta - r_i\cos(\phi_i - \phi))\right] \tag{9}
$$

Finally, we note cos is a symmetric function about the origin, i.e. $\cos(\phi - \phi_i) = \cos(\phi_i - \phi)$. Finally we can return to equation 1 using the relation expressed in equation 9

$$\begin{aligned}
\delta t^{ij}_{expected} &= \frac{1}{c}\left(\vec{w} \cdot \vec{x_i} - \vec{x_j}\right) \\
&= \frac{1}{c}\cos\theta\left[(z_j \tan\theta - r_j \cos(\phi - \phi_j)) - (z_i \tan\theta - r_i \cos(\phi - \phi_i))\right]
\end{aligned} \tag{10}$$

which is where equation 8.3 comes from :)